Experiment 14

LUMPED-PARAMETER DELAY LINE

INTRODUCTION

In this experiment you will explore the properties of a cavity resonator constructed from a dispersive transmission line. This ***lumped parameter delay line*** is made up of 10 concatenated copies of the ***unit cell*** shown in figure 1. To proceed with the analysis of the resonator we must first derive the propagation characteristics of the transmission line; to understand the following discussion you must study ***General Appendix A: Transmission Line Resonance due to Reflections***, which may be found at the end of the lab manual (following Experiment 27).

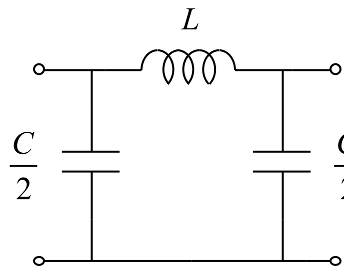

Figure 1: A *unit cell* of the lumped parameter delay line. Several of these cells are connected together to form a transmission line, which could be termed an *LC ladder*. Adjacent capacitors in two connected cells are evidently in parallel, which is equivalent to a single capacitor with value *C*. This is how the actual line is constructed, except for capacitors of value *C*/2 at the two ends of the line. The lab units have $L = 5$ mH and $C = 1$ nF .

DISPERSION RELATION

The transmission line resonator will be of unit length (as in *General Appendix A*) and will be comprised of *N* copies of our unit cell shown in figure 1. For this analysis we will consider the line to be lossless, so the inductors and capacitors making up the line are assumed to be ideal. To derive the ***dispersion relation*** $\omega(k)$ for waves propagating on the line, we use Ohm's law to relate the voltages at adjacent nodes (connections between unit cells; refer to figure 2).

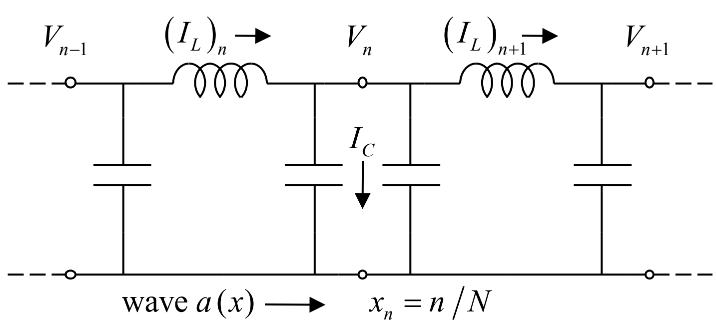

Figure 2: Definitions of the various voltages and currents used to derive the dispersion relation. A wave $a(x)$ propagates to the right. The voltage V_n is at the node connecting the nth and $(n+1)$th unit cells. The current I_C is the sum of the two identical currents through the two parallel unit cell capacitors. The line has unit length, and the total number of unit cells is *N*.

With voltages and currents as defined in figure 2, we see that

$$\begin{gathered} j\omega L\left(I_L\right)_n = V_{n-1} - V_n ; \quad j\omega L\left(I_L\right)_{n+1} = V_n - V_{n+1} ; \\ \left(I_L\right)_n - \left(I_L\right)_{n+1} = I_C = j\omega C V_n \end{gathered} \tag{1}$$

Since the line is assumed to be lossless, the right-going wave $a(x)$ has a propagator $\mathcal{P}(x) = \exp(-jkx)$ (see *General Appendix A*), so the voltages are also related by

$$V_{n-1} = \mathcal{P}(-\Delta x)V_n = V_n e^{-jk(-\Delta x)}; \quad V_{n+1} = \mathcal{P}(+\Delta x)V_n = V_n e^{-jk(+\Delta x)}; \quad \Delta x = 1/N \tag{2}$$

Combining (1) and (2) to find the required relationship between ω and k gives

$$j\omega L\left(I_L\right)_n - j\omega L\left(I_L\right)_{n+1} = V_{n-1} - 2V_n + V_{n+1}$$

$$\therefore\ V_{n-1} - 2V_n + V_{n+1} = j\omega L I_C = \left(j\omega L\right)\left(j\omega C\right)V_n = -\frac{\omega^2}{\omega_{LC}^2} V_n$$

$$\left(e^{jk/N} + e^{-jk/N} - 2\right) V_n = -\frac{\omega^2}{\omega_{LC}^2} V_n$$

$$2 - 2\cos\left(\frac{k}{N}\right) = 4\sin^2\left(\frac{k}{2N}\right) = \frac{\omega^2}{\omega_{LC}^2}$$

$$\boxed{\begin{gathered} \omega\left(k\right) = 2\omega_{LC}\sin\left(\frac{k}{2N}\right) = \omega_c \sin\left(\frac{k}{2N}\right) \\ \omega_{LC} \equiv \frac{1}{\sqrt{LC}}; \quad \omega_c = 2\,\omega_{LC} \end{gathered}} \tag{3}$$

where the total number of unit cells is *N*, and we identify the ***cutoff frequency*** ω_c (recall that our unit of length is the total length of the line, so the wave number *k* is the total radians of phase along the entire line length). Note that ω_c doesn't depend on *N*, the number of unit cells, but only on the characteristics of the unit cell. In fact, it is evident that ω_c is just the resonant frequency of the *L* and *C* around the loop in the unit cell (figure 1) — since the two capacitors are in series (for currents flowing around the loop), the equivalent capacitance is $C/4$, giving a resonant frequency of $1\big/\sqrt{LC/4} = \omega_c$.

Equation (3) is the *dispersion relation* for this transmission line. It describes how the wavelength and frequency are related for a wave propagating on the line. Since the ***phase velocity*** $v_\phi = \omega/k$ varies with frequency, the line is ***dispersive***, and, according to (3), lower frequency waves have a greater phase velocity than high frequency waves. This functional relation $\omega(k)$ is plotted in figure 3.

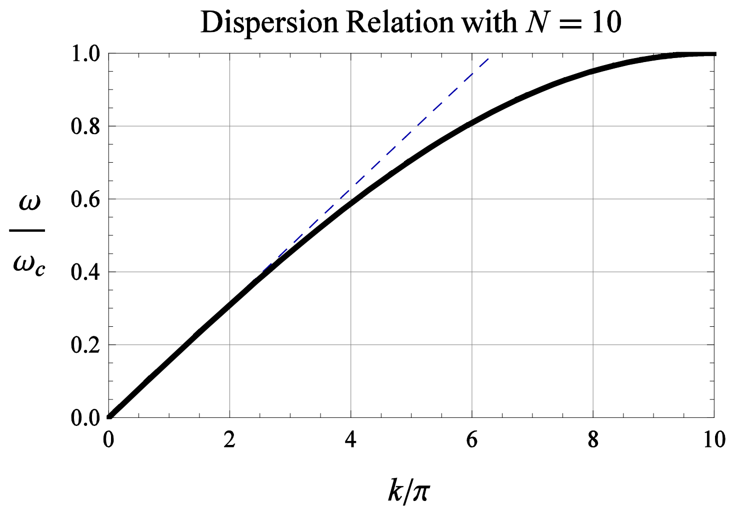

Figure 3: A plot of the dispersion relation (3) for a line with 10 cells and length ≡ 1. The dashed line is the asymptotic, nondispersive relation at low frequencies: $\omega \approx \omega_{LC}\, k / N$. Note that the ***group velocity*** $d\omega/dk$ vanishes at the cutoff frequency $\omega_c = 2\omega_{LC}$, where we also have $k = N\pi$.

The phase delay (phase shift) across a single unit cell is $\Delta\phi = k\,\Delta x = k / N$, so the dispersion relation gives

$$\begin{gathered}\text{phase delay across a single unit cell:}\\ \boxed{\Delta\phi = 2\sin^{-1}\left(\omega/\omega_c\right)}\\ \text{propagation time delay across a single unit cell:}\\ \boxed{\Delta t = \Delta\phi/\omega}\end{gathered} \tag{4}$$

At low frequencies ($\omega \ll \omega_c$) the sine function in (3) is approximately equal to is argument, and the line is approximately nondispersive with phase velocity:

$$\begin{gathered}\text{Low frequency phase velocity } \left(\omega \ll \omega_c\right)\text{:}\\ \boxed{v_\phi \approx \omega_{LC} \quad \langle\text{unit cells/sec}\rangle}\end{gathered} \tag{5}$$

CHARACTERISTIC IMPEDANCE

The other important defining characteristic of the transmission line is its ***characteristic impedance Z_0***, defined in *General Appendix A*. To determine Z_0 we must derive the relation between the voltage at a node joining two unit cells and the current flowing from one cell to the next. As before, consider a line with a right-going wave passing a node in the line (figure 4).

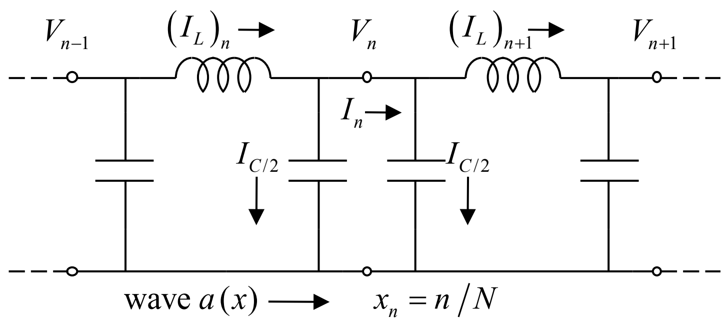

Figure 4: A slight modification of the schematic in figure 2 defines the current I_n flowing through the wire joining the nth and $(n+1)$th unit cells. Instead of the single current I_C there are now two identical currents $I_{C/2}$ flowing through the two parallel unit cell capacitors, each with value $C/2$. The voltage V_n is at the node connecting the two unit cells. As in figure 2, a wave $a(x)$ propagates to the right. The *characteristic impedance* of the transmission line is $Z_0 = V_n / I_n$.

The derivation of Z_0 is thus:

$$\begin{aligned}
I_n &= (I_L)_n - I_{C/2}\,; \quad I_n = (I_L)_{n+1} + I_{C/2} \\
2I_n &= (I_L)_n + (I_L)_{n+1} \\
2I_n &= \frac{V_{n-1} - V_n}{j\omega L} + \frac{V_n - V_{n+1}}{j\omega L} \\
2j\omega L\, I_n &= V_{n-1} - V_{n+1} = \left(e^{jk/N} - e^{-jk/N}\right) V_n \\
\omega L\, I_n &= \sin(k/N)\, V_n
\end{aligned}$$

$$\text{thus, } Z_0 = \frac{\omega L}{\sin(k/N)}$$

To proceed we note that $\sin(k/N) = 2\sin(k/2N)\cos(k/2N)$, so that

$$Z_0 = \frac{\omega L}{2\sin(k/2N)\cos(k/2N)} = \frac{\omega L}{2\sin(k/2N)\sqrt{1-\sin^2(k/2N)}}$$

$$Z_0 = \frac{\omega L}{(\omega/\omega_{LC})\sqrt{1-(\omega/\omega_c)^2}}$$

$$\therefore \boxed{Z_0 = \frac{\sqrt{L/C}}{\sqrt{1-(\omega/\omega_c)^2}}} \tag{6}$$

where we've used the dispersion relation (3) to simplify the final result. At low frequencies $(\omega \ll \omega_c)$ $Z_0 \approx \sqrt{L/C}$, but $Z_0 \to \infty$ as $\omega \to \omega_c$.

CUTOFF FREQUENCY

Now to discuss the significance of the *cutoff frequency* ω_c. Consider the phase delay expression (4). As $\omega \to \omega_c$ from below, the phase delay $\Delta\phi \to \pi$, and $\Delta\phi(\omega_c) = \pi$, so at ω_c the voltage at each node is 180° out of phase with the voltages at the two adjacent nodes. What happens at frequencies $\omega > \omega_c$? In this case the argument of the arcsine in (4) is greater than 1, so the result must be a complex number. We can extend the dispersion relation for frequencies above cutoff by demanding that: (1) ω is real, and (2) the dispersion relation is continuous at $\omega = \omega_c$. Here is the derivation, starting with (3):

$$1 \;\le\; \frac{\omega}{\omega_c} \;=\; \sin\left(\frac{k}{2N}\right) \;=\; \sin\left(A - jB\right)$$

$$\sin\left(A - jB\right) \;=\; \underbrace{\sin A \cosh B}_{\text{real part}} - j\,\underbrace{\cos A \sinh B}_{\text{imaginary part}}$$

$$\therefore \quad \left.\begin{aligned} \cos A \sinh B \;&\equiv\; 0 \\ \lim_{B\to 0}\left(A - jB\right) \;&=\; \pi/2 \end{aligned}\right\} \;\Rightarrow\; A \;=\; \pi/2$$

$$\therefore \quad \frac{\omega}{\omega_c} \;=\; \cosh B\,; \qquad \frac{k}{2N} \;=\; \frac{\pi}{2} - jB$$

$$\therefore \quad \boxed{\text{for } \omega > \omega_c: \quad \frac{k}{2N} \;=\; \frac{\pi}{2} \;-\; j\cosh^{-1}\frac{\omega}{\omega_c}} \qquad (7)$$

This result leads to the following expression for the voltage as a function of position on the line for a "right-going wave"

$$x_n = n/N; \quad -jkx_n \;=\; -j(k/2N)2n \;=\; -j\pi n - 2n\cosh^{-1}\left(\omega/\omega_c\right)$$

$$\boxed{\begin{aligned} &\text{for } \omega > \omega_c: \; \frac{V_n}{V_{n-1}} \;=\; e^{-jk/N} \;=\; -\exp\left[-2\cosh^{-1}\left(\omega/\omega_c\right)\right] \\ &\text{for } \omega \gg \omega_c: \; \frac{V_n}{V_{n-1}} \;\approx\; -\left(2\omega/\omega_c\right)^{-2} \;=\; -\left(\omega/\omega_{LC}\right)^{-2} \end{aligned}} \qquad (8)$$

and we see that ***beyond cutoff frequency there is no wave propagation, but rather the voltage phasor changes sign from node to node and decays geometrically with distance down the line***. We chose $(A - jB)$ in the derivation of (7) to ensure the geometric *decay* of V with distance, rather than its growth.

The characteristic impedance beyond cutoff frequency can be derived almost immediately from equation (6) and is clearly imaginary. We must be careful to choose the proper sign of j in the result; this is done by considering the value of $\sin(k/N)$ in the derivation of (6), given our result (7). The value of Z_0 beyond cutoff is thus

$$\boxed{\text{for } \omega > \omega_c: \quad Z_0 = -j\frac{\sqrt{L/C}}{\sqrt{(\omega/\omega_c)^2 - 1}}}$$
$$\text{for } \omega \gg \omega_c: \quad Z_0 \approx \frac{1}{j\omega}\left(\omega_c\sqrt{L/C}\right) = \frac{1}{j\omega(C/2)} \tag{9}$$

Again, ***no wave propagation is possible beyond cutoff frequency because the characteristic impedance is imaginary***. The impedance beyond cutoff is capacitive; this should also be apparent if you consider the unit cell in figure 1 — clearly at very high frequencies the current entering the unit cell will flow through the nearest parallel capacitor (with value $C/2$).

PROPAGATION ON THE LINE AND REFLECTIONS AT TERMINATIONS

Armed with the dispersion relations (3) and (8) and the characteristic impedance (6) and (9), we are ready to consider the propagation of signals on the transmission line and their reflections from terminations. We know from our reading of *General Appendix A* that the ***voltage reflection coefficient* Γ** from a terminator with impedance Z is given by

$$\boxed{\Gamma = \frac{Z - Z_0}{Z + Z_0}} \tag{10}$$

$$\begin{aligned} Z &= 0 &\Rightarrow\quad \Gamma &= -1 &&\text{(shorted)} \\ Z &= \infty &\Rightarrow\quad \Gamma &= +1 &&\text{(open)} \\ Z &= Z_0 &\Rightarrow\quad \Gamma &= 0 &&\text{(properly terminated)} \end{aligned}$$

The apparatus for this experiment allows you to adjust the termination impedances at both ends of the line. From the discussion in *General Appendix A* we know that the terminated transmission line will be an efficient cavity resonator if the terminations have $|\Gamma| \sim 1$.

Consider the *time domain* behavior of the terminated line in response to a step in the input voltage (a diagram of the configuration for this analysis is shown in figure 5). By applying a low-frequency square-wave input using the signal generator, a series of independent voltage steps are injected onto the transmission line so that its propagation properties and the terminators' refection properties may be studied.

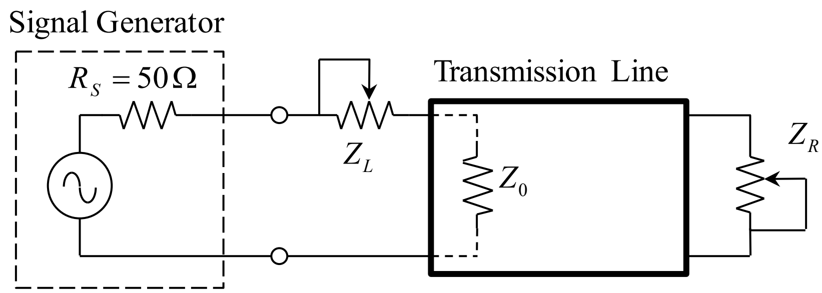

Figure 5: The transmission line is terminated at each end with an adjustable impedance, Z_L on the left (source) end and Z_R on the right end. The signal generator acts as an ideal voltage source in series with a 50Ω resistor, R_S (which is much smaller than the characteristic impedance Z_0). Applying steady sinusoidal signals of various frequencies can excite the many resonances of the line; applying a low-frequency square wave introduces a series of nearly independent voltage step inputs whose propagation down the line and reflections from the terminators may be studied.

Assume that the signal generator in figure 5 is used to inject a single voltage step at time $t = 0$; the voltage step is from an initial value of -1V to a final value of $+1\text{V}$ at the terminals of the signal generator. Assume that Z_L has been set to the value $\sqrt{L/C}$, which is very nearly equal to Z_0 for frequencies below the cutoff frequency, ω_c. Since Z_L and Z_0 form a voltage divider (with $Z_L \cong Z_0$), the voltage step applied to the transmission line is nominally from $-1/2\,\text{V}$ to $+1/2\,\text{V}$, and it is this smaller step which introduces waves at the left end of the line which then propagate toward the termination Z_R.

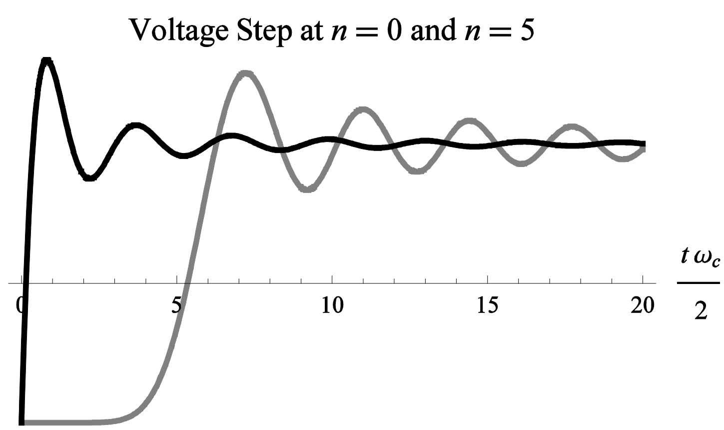

Figure 6: A plot of the step response of the transmission line. The dark line is the response at the left end of the line immediately following a step input by the signal generator (figure 5); the gray line is the response 5 unit cells down the line as the step propagates by that position. As mentioned in the text, the ripples in the response waveforms are caused by the varying characteristic impedance with frequency, $Z_0(\omega)$. The response is less sharp at positions further down the line because the phase velocity is slower for the higher-frequency components of the step. The time axis is in units of the low-frequency propagation time delay/cell, equations (4) and (5).

Figure 6 shows the results for the voltage waveforms at the input to the transmission line and at a position 5 units cells down the line. The Fourier transform of a voltage step consists of a continuously infinite set of sine waveforms, all passing through 0 at time $t = 0$. The amplitudes of the sines are inversely proportional to their frequencies, so all of these waveforms have the same slope (dV/dt) at $t = 0$ (these slopes all add to create the voltage discontinuity represented by the step). If our experimental system is linear, then the evolution of each of these individual sines is independent of all the others; at any point on the line at any later time we can just integrate the instantaneous voltages of all the evolved waves (each at a different frequency) to determine the voltage at that point due to the propagation of the original step input. This process seems pretty complicated (and it can be!), but luckily we can use a tool like *Mathematica*® to do the tedious algebra and numerical integration. Because Z_0 varies with frequency (becoming capacitive above ω_c), the voltage divider formed by Z_L and Z_0 changes the relative amplitudes and phases of the various sines making up the original voltage step. As a result, the waveform at the input to the transmission line is not a perfectly-sharp step, but has ripples, as shown in figure 6. Each of the various waves then propagates down the line at its own phase velocity. The dispersion in phase velocities delays the higher-frequency components more than the low frequencies, so the step becomes ever more spread-out as it propagates down the line (figure 6).

Note that until the signal has had time to propagate down the line to the right-hand termination and back again, the transmission line behaves as though it extends to infinity, because there can be no reflected wave to modify the response. Now consider the effect of an open or of a shorted termination at the right-hand end. The open termination has $\Gamma = +1$, so the reflected step has the same sign as the incoming step. The shorted termination has $\Gamma = -1$, so it inverts the incoming step. The resulting waveforms are shown in figure 7.

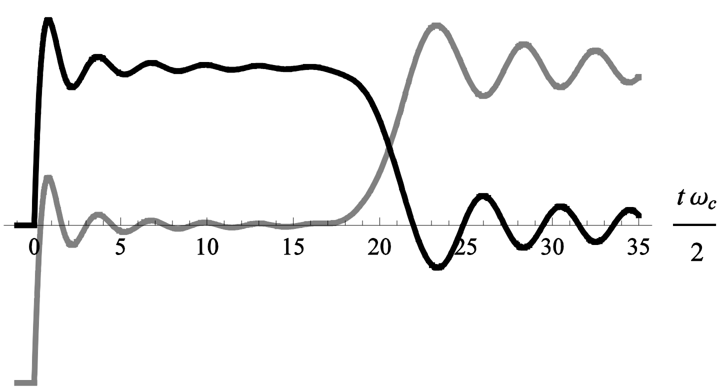

Figure 7: arrival of a reflection from the right-hand end of the line. Both plots show the waveform at the input end of the transmission line: the dark line is the response with the right-hand end shorted ($\Gamma = -1$), so the reflection is inverted; the gray line is the response with the right-hand end open ($\Gamma = +1$), so the reflection reinforces the original step. The length of the line is 10 unit cells, so the reflection arrives 20 time units after the initial stimulus. The source end of the line is properly terminated, so no further reflection takes place at that end. The final, equilibrium voltage on the line ($t \to \infty$) is 0 for the shorted case and 1 for the open case (signal generator output = –1 to +1).

STEADY-STATE RESPONSE AND RESONANCES

General Appendix A provides a brief introduction to the topic of cavity resonance and derives the conditions on the wave number (k) for resonance to occur, depending on the character of the cavity terminations. These conditions are summarized here:

Same termination at both ends (both shorted, both open):

$$\boxed{k_m = m \times \pi} \qquad m \in \mathbb{N} \tag{11}$$

Opposite terminations at the ends (one shorted, one open):

$$\boxed{k_m = \left(m - \tfrac{1}{2}\right) \times \pi} \qquad m \in \mathbb{N} \tag{12}$$

where, as in *General Appendix A*, k_m is the total phase along the line for the m-th resonance (or, equivalently, the line is taken to have unit length, and k_m is the wave number). ***So for the same termination at both ends, at a resonance the transmission line is an even number of ¼ wavelengths long; if the terminations are different, then there are an odd number of ¼ wavelengths at resonance.*** This condition ensures that the reflected wave, once it has been reflected by both ends of the line, is again in phase with the original wave.

There is another interesting interpretation of these resonance conditions: if the far end of the transmission line is shorted, say, then whenever the line length is an even number of ¼ wavelengths the line presents a short circuit to the source driving it; it presents an open circuit whenever the length is an odd number of ¼ wavelengths. These results would be the other way around if the far end of the line is open-circuited.

For a given configuration of end terminations, ***the number of distinct resonances is the same as the total number N of unit cells making up the line***. This result is relatively easy to derive and is left to the problems. This result also follows by considering the number of dynamical degrees of freedom on the transmission line. There are N degrees of freedom, one for each unit cell (the instantaneous dynamical state of a cell could be specified, for example, by specifying the current flowing around the loop formed by the inductor and the two capacitors of the cell and its time derivative [see figure 1]). As you will learn in your advanced classical mechanics course, the number of *normal modes* of a dynamical system is the same as its number of degrees of freedom; as in other similar systems, its normal modes are just the resonant modes.

PRELAB PROBLEMS

1. The phase velocity is $v_\phi(\omega) = \omega / k$. Use the dispersion relation (3) to show that:

$$\frac{v_\phi(\omega \ll \omega_c)}{v_\phi(\omega_c)} = \frac{\pi}{2}$$

2. Show that if the total number of unit cells is *N*, then this is also the number of discrete resonant frequencies of a cavity constructed from the transmission line, as stated in the discussion following equation (12). Consider both end termination cases, equations (11) and (12). Hint: each resonant frequency must have $\omega \le \omega_c$, so that $\sin(k/2N) \le 1$ (equation (3)).

3. If $\boxed{L = 5.0\,\text{mH and } C = 1.0\,\text{nF}}$, then what are Z_0, ω_c, and $f_c = \omega_c / 2\pi$? What would be the low-frequency ($\omega \ll \omega_c$) phase velocity in unit cells/sec?

4. Given the results of the previous problem, and if the transmission line has 10 unit cells ($N = 10$), what should be the lowest resonant frequency for each of the following termination conditions:

   a. Both ends shorted
   b. Source end shorted, far end open

   Sketch the amplitude v. position, $V(x)$, at that resonant frequency for each of the above cases.

PROCEDURE

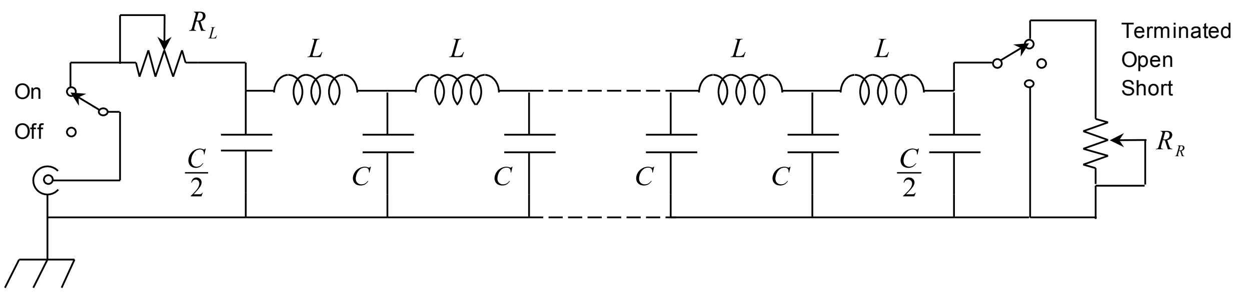

Figure 8: The lumped-element transmission line apparatus consists of $N = 10$ unit cells joined as shown. Two variable resistors, R_L on the left (source) end and R_R on the right end, permit adjustment of the reflection coefficients of the line terminations. A switch on the right end allows an additional selection of shorted or open termination at that end. The signal generator is connected to the BNC connector on the left end of the line; it may be isolated from the circuit using the other switch so that the resistor values may be measured without damaging an ohmmeter. $L = 5.0\,\text{mH}$, $C = 1.0\,\text{nF}$, R_L and R_R are adjustable 0–20 kΩ. Once the signal generator is attached, the bottom conductor will be connected to ground, as shown.

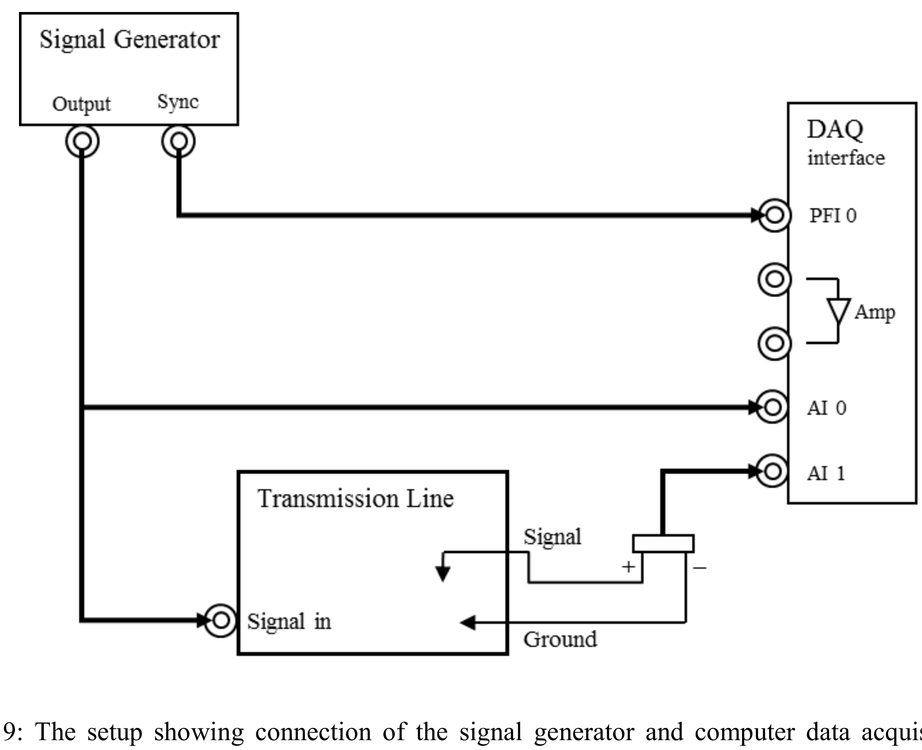

Figure 9: The setup showing connection of the signal generator and computer data acquisition (DAQ) hardware. Not shown is the shorting jumper wire used to short the resistance of R_L (figure 8) during steady-state cavity resonance measurements. Computer data acquisition and control of the experiment is similar to that of Experiment 2. The AI 1 signal lead (+) may be connected to the various nodes joining unit cells; the (–) input should be connected to the common ground conductor.

Ensure that the apparatus has been assembled as in figures 8 and 9. Familiarize yourself with the delay line assembly and its switches and variable resistors. The investigations to be completed during the experiment are:

1. Investigate the propagation and reflection of a step input.
2. Determine the characteristic impedance (in the low-frequency limit).
3. Set-up and measure the lowest resonant frequency for the shorted–open line.
4. Examine the shape of the wave on the line at this resonant frequency.
5. Measure the detailed frequency response of this cavity resonator over a wide range of frequencies. Determine the cutoff frequency.
6. Investigate the behavior of the transmission line at frequencies beyond the cutoff frequency.
7. Find the self-resonance frequency of the inductors used in the transmission line.
8. (optional) Measure the frequency response at node 9 of the shorted–shorted configuration.

What follow are additional notes and guidance for completing these investigations.

Use the *Transient Response* program and input a low-frequency (~200 Hz?) square-wave to investigate the propagation properties of the transmission line and the reflections produced by various settings of the termination impedances. Examine the signal at the left end of the transmission line (node 0). With the right termination set to *Short* and then *Open*, compare the signal with that predicted by the theory (figure 7).

Proper adjustment of the source termination (R_L) should result in only a single refection; if it is not adjusted to $\sqrt{L/C}$, you should see several reflections of decreasing amplitude, each arriving after another round-trip delay of the transmission line. When R_L is set properly, the initial step amplitude at node 0 should also have approximately ½ the amplitude of the square-wave input step (AI 0).

How is the round-trip delay time for the reflection related to the lowest resonant frequency (both ends shorted)? What is the low-frequency delay time per unit cell? How does this compare to the low-frequency phase velocity you calculated (problem 3)?

Next set the right-end switch to *Terminated* and adjust R_R until the reflection vanishes, properly terminating both ends of the line. Note the shape of the signal at node 0 and its relation to the input square-wave from the signal generator. Does its amplitude match the theory? Look at the signal at node 5 and compare it to figure 6.

Turn the left switch to *Off*, disconnecting the signal generator, before you measure R_L and R_R using an ohmmeter. How does the 50Ω output impedance of the signal generator affect the value of R_L when properly set to eliminate reflections?

For the cavity frequency response investigations, short R_L using a jumper wire so that the signal generator and AI 0 are connected directly to the input of the transmission line. Connect AI 1 at the other end of the line with the termination switch to *Open*. Use a small-amplitude sine-wave input from the signal generator to find the first resonant frequency for this configuration (shorted–open). *Make sure your input signal amplitude is not set too high!* The phase at the lowest resonance should be −90°, and the gain is $(4/\pi)Q$ (*General Appendix A* equations (22) and (19), with $k_n = \pi/2$ and $x = 1$).

Use the *Frequency Response* program to accurately find the lowest resonant frequency. Note the gain and phase at the resonance (this would be $V(1)/V(0)$ using the notation of *General Appendix A*). Record the gain and phase at each of the nodes 1 to 10 for later analysis, so that you can compare to *General Appendix A* equation (22) for $V(x)/V(0)$ at a resonance (remember, *x* goes from 0 at left end to 1 at the right end of the line, so $x = n/N$ for node *n*).

Since the right end of the line is an anti-node for all resonances in the shorted–open configuration, you should connect AI 1 at that location for your frequency sweeps (up to and including the cutoff frequency).

At some frequency slightly above cutoff, you should record the gain and phase for each of the first few nodes to compare with the theory (equation (8)). Frequency sweeps above cutoff should be done with AI 1 connected at node 1. Find the inductor self-resonance frequency by looking for a sharp null in the output at node 1 at a frequency a few times higher than the cutoff frequency (refer to figure 12 in Appendix A of these notes, pg 14–16). The phase is 180° below this frequency and 0° above it.

DATA ANALYSIS

Do the step-input observations match the theory presented on pages 14–6 to 14–8? Compare the low-frequency phase velocity to ω_{LC}.

Do the gain and phase v. position data (procedure step 4) match the theory (*General Appendix A* equation (22))?

Assume the wave number for each of the shorted–open resonances is that given in equation (12). Calculate $x_m \equiv \sin(k_m/2N)$ for each of the resonances and fit this *modified* dispersion relation ω_m v. x_m using your observed resonant frequencies. According to the theory (equation (3)), should the fit be a strict proportion ($\omega_m = \omega_c x_m$)? What does this fit give for the cutoff frequency, ω_c? Is there a pattern to the residuals?

Transform your data to $(1/\omega_m)^2$ v. $(1/x_m)^2$ and try a linear fit, that is: $(1/\omega_m)^2 = a + b(1/x_m)^2$. Is the fit better? Compare $2\omega_{LC}$ to $b^{-1/2}$ and your observed inductor self-resonance frequency, ω_s, to $a^{-1/2}$. For an extension to the theory which includes ω_s see Appendix A of this experiment.

APPENDIX A: EFFECTS OF INDUCTOR SELF-RESONANCE

An inductor in the lumped-parameter transmission line is made by winding a long, thin wire into a tightly-wrapped coil and inserting it into a donut-shaped ferrite form. Because the many windings of the coil are very close to one another and are separated by only a thin layer of insulation, electric fields form between adjacent windings and attract charges in the wire. In other words, adjacent windings form little capacitors which provide an alternate path for the varying current flow through the wire. Consequently, the coil of wire has a capacitance which is situated in parallel with the inductance resulting from current flow which follows the turns in the coil. The equivalent circuit of the coil which includes this capacitance is shown in figure 10.

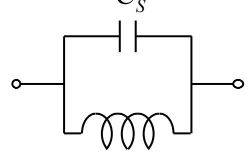

Figure 10: The equivalent circuit of an inductor formed from a closely-packed coil of wire includes a parallel capacitance, shown here with value C_S. The impedance of the coil is then the parallel combination of the two component impedances, $j\omega L$ and $1/j\omega C_S$. The parallel L and C_S form a "tank circuit."

The parallel combination of the inductor and capacitor has an equivalent impedance which becomes infinite at the *self-resonance frequency* of the coil, $\omega_s = 1/\sqrt{LC_S}$; this impedance is:

$$\begin{aligned} Z_{coil} &= \left(j\omega L\right) \| \left(1/j\omega C_S\right) \\ &= \frac{j\omega L}{1-\left(\omega/\omega_s\right)^2} = \frac{1}{j\omega C_S}\,\frac{1}{1-\left(\omega_s/\omega\right)^2} \end{aligned} \tag{13}$$

So for frequencies $\omega < \omega_s$ the coil behaves as an inductor, but for frequencies $\omega > \omega_s$ it has a capacitive impedance. As seen in (13), $Z_{coil} \to \infty$ at ω_s. To include this effect in the theory of the transmission line's dispersion relation, we substitute Z_{coil} for $j\omega L$ in the derivation leading to equation (3) on page 14–2:

$$V_{n-1} - 2V_n + V_{n+1} = Z_{coil}\, I_C = Z_{coil}\left(j\omega C\right)V_n = -\frac{\omega^2}{\omega_{LC}^2} \times \frac{1}{1-\left(\omega/\omega_s\right)^2} V_n$$

$$e^{jk/N} + e^{-jk/N} - 2 = -\frac{\omega^2}{\omega_{LC}^2} \times \frac{1}{1-\left(\omega/\omega_s\right)^2}$$

$$\therefore\quad 4\,\omega_{LC}^2 \sin^2\!\left(\frac{k}{2N}\right) = \frac{\omega^2}{1-\left(\omega/\omega_s\right)^2} = \frac{1}{\left(1/\omega\right)^2 - \left(1/\omega_s\right)^2}$$

$$\boxed{\frac{1}{\omega^2} = \frac{1}{\omega_s^2} + \frac{1}{4\,\omega_{LC}^2 \sin^2\left(k/2N\right)}} \tag{14}$$

Equation (14) is the new dispersion relation which includes the effects of inductor self-resonance. Cutoff still occurs when the argument of the sine is $\pi/2$; the group velocity vanishes here, and k must become complex at higher frequencies. The cutoff frequency, however, is clearly different from that defined by the original dispersion relation (3). When the sine in (14) is 1, the modified cutoff frequency is lower; see (15) and figure 11, below.

$$\boxed{\omega_c = \frac{2\omega_{LC}}{\sqrt{1+4\omega_{LC}^2/\omega_s^2}} = \frac{1}{\sqrt{L\left(C/4+C_S\right)}}} \tag{15}$$

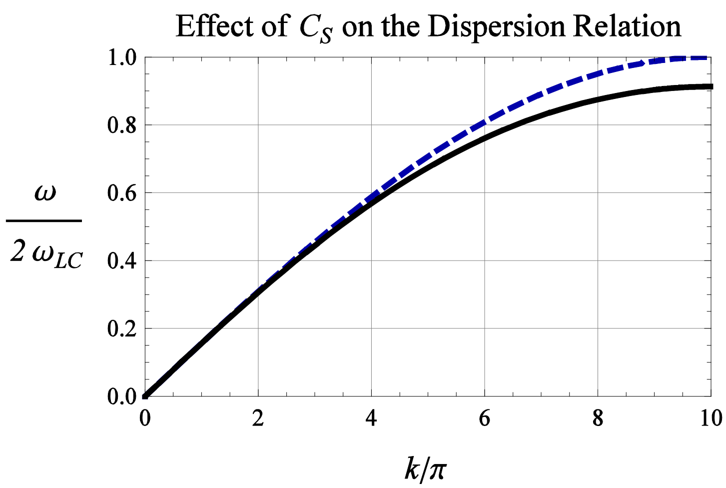

Figure 11: A comparison of the dispersion relation (3) (dashed) with (14) (solid) for $C = 20\times C_S$. The cutoff frequency ω_c is reduced by ~10% by the presence of C_S. The wave number at cutoff is $k = N\pi$ for each form of the dispersion relation.

The last expression in (15) shows that, again, ω_c is just the resonant frequency of the L and total equivalent C in the circuit loop of a unit cell — the coil's stray capacitance C_S is in parallel with the other capacitors, so its capacitance adds to their equivalent series combination, $C/4$.

When $\omega > \omega_c$, the wave number k must be complex, as we found previously, so waves do not propagate on the line at frequencies above cutoff. For $\omega_c < \omega < \omega_s$ the behavior is similar to that of equation (8), so there is a 180° phase shift from node to node, and the amplitude falls geometrically with node number:

$$\boxed{\text{for } \omega_c < \omega < \omega_s: \quad \frac{V_n}{V_{n-1}} = -\exp\left[-2\cosh^{-1}\left(\frac{\omega}{2\omega_{LC}\sqrt{1-\left(\omega/\omega_s\right)^2}}\right)\right]} \tag{16}$$

From the above expression it is clear that $V_n = 0$ at $\omega = \omega_s$, as mentioned before. This fact provides the most straightforward method to experimentally identify ω_s. For frequencies above ω_s, the above expression must be modified. Without proof, we state the behavior in (17).

$$
\boxed{\begin{aligned}
\text{for } \omega > \omega_s: \quad \frac{V_n}{V_{n-1}} &= \exp\left[-2\sinh^{-1}\left(\frac{\omega}{2\omega_{LC}\sqrt{(\omega/\omega_s)^2-1}}\right)\right] \\
\text{for } \omega \gg \omega_s: \quad \frac{V_n}{V_{n-1}} &\approx \left(1+\frac{\omega_s^2}{2\omega_{LC}^2}+\sqrt{\frac{\omega_s^2}{\omega_{LC}^2}+\frac{\omega_s^4}{4\omega_{LC}^4}}\right)^{-1} \\
&\approx \left(\frac{C_S}{C+C_S}\right); \quad C \gg C_S
\end{aligned}} \tag{17}
$$

Above ω_s the phase shift is 0 (all nodes are in phase), and as ω continues to increase, the attenuation per cell approaches a constant value, which is just the attenuation due to a capacitive voltage divider ladder consisting of capacitors C_S and C (figure 12).

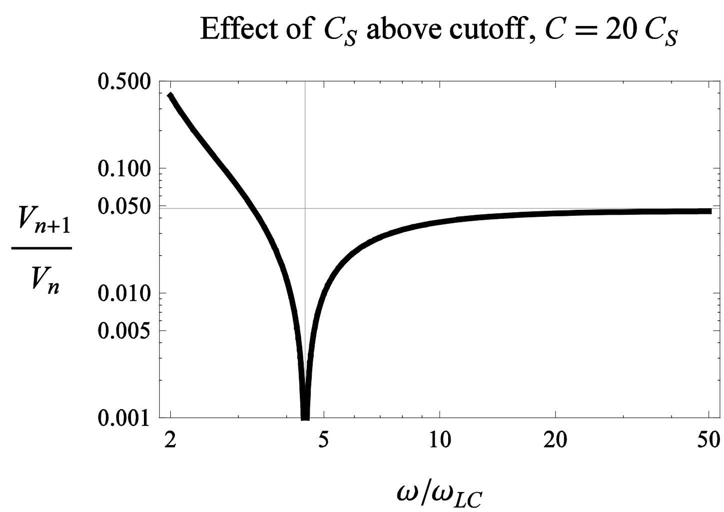

Figure 12: The attenuation of the input signal above cutoff frequency, including the effect of the inductor's stray capacitance, C_S. The vertical gridline is at ω_s; the horizontal gridline at the capacitive voltage divider ratio given by the final expression in (17).